DERIVING ANGULAR MOMENTUM SUM RULES

THE GOOD, THE BAD AND THE UGLY

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Background to the study—or why did we bother to work like slaves for several months?

Shore and White's, at first sight, surprising claim about the axial anomaly.

Based on a classical paper of Jaffe and Manohar who stressed the subtleties and warned that 'a careful limiting procedure has to be introduced'.

Despite all the care, there are flaws. With the J-M result one cannot have a sum rule for a transversely polarized nucleon.

With the correct version one can!

OUTLINE OF TALK

The Ugly: The traditional way of deriving angular momentum sum rules. Its pitfalls and problems. Horrible infinities all over the place.

The Bad: Our improvement of the traditional approach. No infinities but the price is high in terms of complexity.

The Good: Larry Trueman's brilliant idea. All is beautiful and simple.

What is the aim???

We consider a nucleon with 4-momentum p^μ and covariant spin vector S corresponding to some specification of its spin state e.g. helicity, transversity or spin along the Z-axis i.e. a nucleon in state $|p, S\rangle$.

We require an expression for the expectation value of the angular momentum in this state i.e. for $\langle p, S|\boldsymbol{J}|p, S\rangle$

i.e. we require an expression in terms of p and S.

Why is this useful?

When we introduce a model of the nucleon it can be used to relate the expectation value of $\boldsymbol{J}$ for the nucleon to the angular momentum carried by its constituents.

But models (e.g. the parton model) are only valid in certain kinematic regimes e.g for partons in the 'infinite momentum frame', which, in practice, means a frame where $E \gg M$ for the nucleon.

So we need an expression for the matrix element valid in any frame!

THE UGLY

The traditional approach: In every field theory there is an expression for the angular momentum density operator. The angular momentum operator $\boldsymbol{J}$ is then an integral over all space of this density.

To understand the subtleties we need to recall Noether's famous theorem:

For every continuous symmetry there is a conserved current and a conserved operator which generates the transformations of that symmetry.

Thus invariance under time translations $\Rightarrow$ conservation of the energy operator (or Hamiltonian) P_0.

Invariance under spatial translations $\Rightarrow$ conservation of linear momentum $\boldsymbol{P}$

Then translations in space-time are generated as follows: For any local operator $F(x)$

$$F(x+a) = e^{iP.a} F(x) e^{-iP.a}$$

Thus

$$F(x) = e^{iP.x} F(0) e^{-iP.x}$$

Danger! $G(x) = xF(x)$ seems like a reasonable local operator.

But by the above:

$$G(x) = e^{iP.x} G(0) e^{-iP.x}$$

$\therefore G(x) = 0$ for ALL x.

Clearly absurd!

Typically the angular momentum density involves the energy-momentum tensor density $T^{\mu\nu}(x)$ in the form e.g.

$$\boldsymbol{J}_z = \boldsymbol{J}^3 = \int dV[xT^{02}(\boldsymbol{x}) - yT^{01}(\boldsymbol{x})]$$

Consider the expectation value of the first term

$$\langle p, \mathsf{S}| \int dV x T^{02}(\boldsymbol{x}) |p, \mathsf{S}\rangle = \int dV x \langle p, \mathsf{S}| T^{02}(\boldsymbol{x}) |p, \mathsf{S}\rangle$$

$$= \int dV x \langle p, \mathsf{S}| e^{i\boldsymbol{P}.\boldsymbol{x}} T^{02}(0) e^{-i\boldsymbol{P}.\boldsymbol{x}} |p, \mathsf{S}\rangle$$

Now the nucleon is in an eigenstate of momentum, so $\boldsymbol{P}$ acting on it just becomes $\boldsymbol{p}$. The numbers $e^{i\boldsymbol{p}.\boldsymbol{x}}e^{-i\boldsymbol{p}.\boldsymbol{x}}$ cancel out and we are left with:

$$\int dVx\langle p, \mathsf{S}|T^{02}(0)|p, \mathsf{S}\rangle$$

The matrix element is independent of x so we are faced with $\int dVx = \infty$? or $= 0$? Totally ambiguous!

The problem is an old one: In ordinary QM plane wave states give infinities

The solution is an old one: Build a wave packet, a superposition of physical plane wave states

In QM we use

$$\Psi_{\boldsymbol{p}_0}(\boldsymbol{x}) = \int d^3\boldsymbol{p}\, \psi(\boldsymbol{p}_0 - \boldsymbol{p})\, e^{i\boldsymbol{p}.\boldsymbol{x}}$$

where $\psi(\boldsymbol{p}_0 - \boldsymbol{p})$ is peaked at $\boldsymbol{p} = \boldsymbol{p}_0$

We then calculate some physical quantity and at the end take the limit of a very sharp wave packet

In field theory we do essentially the same and build a physical wave packet state:

$$|\Psi(\boldsymbol{p}_0)\rangle = \int d^3\boldsymbol{p}\, \psi(\boldsymbol{p}_0 - \boldsymbol{p})\, |\boldsymbol{p}\rangle$$

then an expectation value in the state $|\Psi(\boldsymbol{p}_0)\rangle$ will involve non-diagonal matrix elements

$$\langle \boldsymbol{p}'|\boldsymbol{J}|\boldsymbol{p}\rangle$$

What about the spin??? J-M use

$$|\Psi(\boldsymbol{p}_0, S)\rangle = \int d^3\boldsymbol{p}\, \psi(\boldsymbol{p}_0 - \boldsymbol{p})\, |\boldsymbol{p}, S\rangle$$

i.e. with a fixed S on both sides of the equation.

They do this to simplify things so that the expectation value only involves

$$\langle \boldsymbol{p}', S|\boldsymbol{J}|\boldsymbol{p}, S\rangle$$

i.e. is at least diagonal in S—important for them because they try to write down the most general form for this matrix element

But this is incorrect. The wave packet is not physical. Recall that for a physical nucleon

$$p.S = 0$$

Thus if $\boldsymbol{p}$ is to vary freely in the wave packet integration S cannot remain fixed. — Point 1

The second difficulty is the general form written down for the matrix element. The Lorentz structure assumed is not correct for non-diagonal matrix elements.

To see this think of electromagnetic form factors:

$$\langle p', S | j^{\mu}_{em} | p, S \rangle$$

We cannot say: this transforms like a 4-vector, therefore we can express it terms of vectors built from p, p', S

We have to first factor out the Dirac spinors

$$\bar{u}(p', S)[\gamma^{\mu} F_1 + \frac{i\sigma^{\mu\nu} q_{\nu}}{2m} F_2] u(p, S)$$

This is the second problem—-Point 2

THE BAD

Correcting the traditional approach

Point 1: $\acute{a}$ la BLT, sandwich $\boldsymbol{J}$ between physical wave packet states

$$|\Psi(\boldsymbol{p}_0, \boldsymbol{s})\rangle = \int d^3\boldsymbol{p}\, \psi(\boldsymbol{p}_0 - \boldsymbol{p})\, |\boldsymbol{p}, \boldsymbol{s}\rangle$$

where $\boldsymbol{s}$ is the spin vector in the rest frame.

Note that the covariant spin vector, for spin quantized along the Z axis, is then

$$S^\mu = \left(\frac{\boldsymbol{p}.\boldsymbol{s}}{m}, \boldsymbol{s} + \frac{\boldsymbol{p}.\boldsymbol{s}}{m(p_0 + m)}\boldsymbol{p}\right)$$

Thus S varies as we integrate over $\boldsymbol{p}$

Point 2: Take care to utilize Dirac spinors when writing down general structure of

$$\langle \boldsymbol{p}', \boldsymbol{s}|\boldsymbol{J}|\boldsymbol{p}, \boldsymbol{s}\rangle$$

•Long, involved calculation!

• Need to study narrow wave packet and limit as it approaches plane wave

Result: For general polarization state of nucleon BLT differs from J-M. Details later.

Obviously, we sent a draft of our paper to Jaffe and Manohar and were relieved and heartened by their positive reaction. And they have kindly given me permission to include a copy of their letter to us. An abridged version follows:

Dear Larry, Elliot, Bernard,

Better late than never. Aneesh and I finally found ourselves in the same place with the time to review the issues you raised by email and in your recent paper. We agree that there is an error in our eq. (6.9). It came from treating the quantity u(p', s)u(p, s) with insufficient care.

Thanks for taking care and finding this mistake. It's good to get it cleared up.

I have to add that I found your paper rather difficult to read. There is quite a bit of stuff that gets in the way of the relatively simple error. For example I'm not sure why you have to go through wave packets any more than we did. We made a rather straightforward error of replacing u(p', s)u(p, s) by unity. Isn't it rather easy to set straight? All the best, Bob and Aneesh

THE GOOD

Larry Trueman's brilliant idea

• It is simple.

• It is short

• It works for any spin. Previous methods only work for spin 1/2.

We know how rotations affect states. If $|\boldsymbol{p}, m\rangle$ is a state with momentum $\boldsymbol{p}$ and spin projection m in the rest frame of the particle, and if $\widehat{R}_i(\beta)$ is the operator for a rotation β about the axis i, then

$$\widehat{R}_i(\beta)|\boldsymbol{p}, m\rangle = |\boldsymbol{R}_i(\beta)\boldsymbol{p}, m'\rangle\, D^s_{m'm}[R_i(\beta)]$$

But rotations are generated by the angular momentum operators! i.e.

$$\hat{R}_i(\beta) = e^{-i\beta \boldsymbol{J}_i}$$

so that

$$\boldsymbol{J}_i = i \frac{d}{d\beta} \hat{R}_i(\beta) \Big|_{\beta=0}$$

From the above we know what the matrix element of $\hat{R}_i(\beta)$ looks like. So we simply differentiate, multiply by i, and put $\beta = 0$.

Thus we have

$$\langle \boldsymbol{p}', m' | \boldsymbol{J}_i | \boldsymbol{p}, m \rangle = i \frac{\partial}{\partial \beta} \langle \boldsymbol{p}', m' | R_i(\beta) | \boldsymbol{p}, m \rangle |_{\beta=0}$$

$$= \quad i \frac{\partial}{\partial \beta} \left[\langle \boldsymbol{p}', m' | R_i(\beta) \boldsymbol{p}, n \rangle \, D^s_{nm}[R_i(\beta)] \right]_{\beta=0}$$

One technical point: you have to know that the derivative of the rotation matrix for spin s at $\beta = 0$ is just the spin matrix for that spin. (more correctly: the matrix generator of rotations for that spin) e.g. for spin 1/2 just $\sigma_i/2$.

COMPARISON OF RESULTS

For the expectation values we find, for any spin configuration (longitudinal, transverse etc) the remarkably simple result (suppressing a delta-function term):

$$\langle\langle \boldsymbol{p}, \boldsymbol{s}|\boldsymbol{J}_i|\boldsymbol{p}, \boldsymbol{s}\rangle\rangle = \frac{1}{2}\boldsymbol{s}_i$$

Written in these variables the J-M result is:

$$\langle\langle \boldsymbol{p}, \boldsymbol{s}|\boldsymbol{J}_i|\boldsymbol{p}, \boldsymbol{s}\rangle\rangle_{JM} = \frac{1}{4mp_0}\left[(3p_0^2 - m^2)\boldsymbol{s}_i - \frac{3p_0 + m}{p_0 + m}(\boldsymbol{p}.\boldsymbol{s})\boldsymbol{p}_i\right]$$

These look completely different. But for a state of longitudinal polarization i.e when $\boldsymbol{s} = \widehat{\boldsymbol{p}}$ they agree!

But for transverse spin they are crucially different.

This difference is critical for the purpose of deriving angular momentum sum rules, because these are derived for a fast moving nucleon i.e. for $p_0 \to \infty$.

For transverse spin i.e. for $\boldsymbol{s}$ perpendicular to $\boldsymbol{p}$ the J-M result gives:

$$\langle\langle \boldsymbol{p}, \boldsymbol{s} | \boldsymbol{J}_i | \boldsymbol{p}, \boldsymbol{s} \rangle\rangle_{JM} = \frac{1}{4mp_0}\left[(3p_0^2 - m^2)\boldsymbol{s}_i\right]$$

which $\to \infty$ as $p_0 \to \infty$, so no sum rule is possible.

SUM RULES

Expand nucleon state as superposition of n-parton Fock states.

$$
\begin{aligned}
|\boldsymbol{p}, m\rangle \simeq \sum_n \sum_{\{\sigma\}} \int d^3\boldsymbol{k}_1 \dots d^3\boldsymbol{k}_n \\
\psi_{\boldsymbol{p},m}(\boldsymbol{k}_1, \sigma_1, \dots \boldsymbol{k}_n, \sigma_n) \\
\delta^{(3)}(\boldsymbol{p} - \boldsymbol{k}_1 \dots - \boldsymbol{k}_n)|\boldsymbol{k}_1, \sigma_1, \dots \boldsymbol{k}_n, \sigma_n\rangle.
\end{aligned}
$$

There are two independent cases:

(a)Longitudinal polarization i.e. $\boldsymbol{s}$ along OZ. The sum rule for $\boldsymbol{J}_z$ yields the well known result

$$
1/2 = 1/2\,\Delta\Sigma + \Delta G + \langle L_z^q \rangle + \langle L_z^G \rangle
$$

(b) Transverse polarization i.e. $\boldsymbol{s} \perp \boldsymbol{p}$. The sum rule for $\boldsymbol{J}_x$ or $\boldsymbol{J}_y$ yields a a new sum rule

$$1/2 = 1/2 \sum_{flavs} \int dx \, [\Delta_T q(x) + \Delta_T \bar{q}(x)] + \sum_{q,\bar{q},G} \langle L_{\boldsymbol{s}_T} \rangle$$

Here $L_{\boldsymbol{s}_T}$ is the component of $\boldsymbol{L}$ along $\boldsymbol{s}_T$.

The structure functions $\Delta_T q(x) \equiv h_1^q(x)$ are known as the quark transversity or transverse spin distributions in the nucleon.

As mentioned no such parton model sum rule is possible with the J-M formula because, as $p \to \infty$, for $i = x, y$ the matrix elements diverge.

It is absolutely crucial to note that the sum rule involves a SUM of Quark and Antiquark densities.

Not realizing this has led to some misunderstandings.

What some people call the TRANSVERSITY of the NUCLEON is the difference between quark and antiquark contributions.

Thus the transverse spin sum rule, although it involves the transverse spin or transversity quark and antiquark densities, does NOT involve the nucleon's transversity. The transversity OPERATOR is NOT related to the angular momentum.

The structure functions $\Delta_T q(x) \equiv h_1^q(x)$ are most directly measured in doubly polarized Drell-Yan reactions

$$p(\boldsymbol{s}_T) + p(\boldsymbol{s}_T) \to l^+ + l^- + X$$

where the asymmetry is proportional to

$$\sum_f e_f^2 [\Delta_T q_f(x_1) \Delta_T \bar{q}_f(x_2) + (1 \leftrightarrow 2)].$$

They can also be determined from the asymmetry in semi-inclusive hadronic interactions like

$$p + p(\boldsymbol{s}_T) \to H + X$$

where H is a detected hadron, typically a pion.

Also in SIDIS reactions with a transversely polarized target

$$\ell + p(\boldsymbol{s}_T) \to \ell + H + X.$$

The problem is that in these semi-inclusive reactions $\Delta_T q_f(x)$ always occurs multiplied by the largely unknown Collins fragmentation function. Moreover recent studies seem to indicate that in hadronic reactions the Collins asymmetry is largely washed out by phase effects.

SUMMARY

• In order to derive angular momentum sum rules you need an expression for the matrix elements of the angular momentum operators $\boldsymbol{J}$ in terms of the momentum $\boldsymbol{p}$ and spin $\boldsymbol{s}$ of the particle.

• Such matrix elements are divergent and ambiguous in the traditional approach and are incorrect in some classic papers

• This can be handled using wave packets but the calculations are long and unwieldy

• Using our knowledge of how states transform under rotations leads quickly and relatively painlessly to correct results

• The great success of the correct approach is that it allows derivation of a sum rule also for transversely polarized nucleons